Name: ____________________________

Score: ________ /20

Line Integral Theorem Practice

Please staple your work and use this page as a cover page.

1. Let $\vec{F}(x,y) = \langle y^2, x^2y\rangle$ and C the rectangle with vertices $(0,0)$, $(5,0)$, $(5,4)$, and $(0,4)$ and positive orientation. Consider the line integral
$$\int_C \vec{F}\cdot d\vec{r}.$$
   (a) Compute this line integral directly by definition.
   (b) Compute the integral instead using Green's Theorem.

2. Compute $\int_C \vec{F}\cdot d\vec{r}$ if $\vec{F} = \langle y\cos x - xy\sin x, xy + x\cos x\rangle$ and C is the triangle formed by going from $(0,0)$ to $(0,4)$ to $(2,0)$ to $(0,0)$.

3. Say that we have a region D in the plane with closed boundary curve C that is positively oriented, $\vec{n}$ the outward unit normal vector to the curve C, and a vector field $\vec{F} = \langle P, Q\rangle$. Green's Theorem can also be written in the following vector form
$$\int_C \vec{F}\cdot\vec{n}\,ds = \iint_D \text{div}\vec{F}\,dA = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\,dA,$$
   Using this version of Green's Theorem, compute $\int_C \vec{F}\cdot\vec{n}\,ds$, if $\vec{F} = \langle xy^2, x^2y\rangle$ and C is the curve given by $\vec{r}(t) = \langle 3\cos t, 3\sin t\rangle$, $0 \leqslant t \leqslant 2\pi$ with positive orientation.

   Note: the quantity div$\vec{F}$ appearing in this integral is called the *divergence* of $\vec{F}$. We will soon see and study some useful properties of the divergence of a vector field.

4. Let $\vec{F}(x,y,z)$ be a vector field and assume $\vec{F} = \vec{\nabla} f$ for some function $f(x,y,z)$. If $\int_{C_1} \vec{F}\cdot d\vec{r} = 2$ and C_1 is given by $\vec{r_1}(t) = \langle t, t^2, t^3\rangle$, $0 \leqslant t \leqslant 1$, then what is the value of $\int_{C_2} \vec{F}\cdot d\vec{r}$, if C_2 is given by $\vec{r_2}(t) = \left\langle 1 - \frac{1}{2}t, -\frac{1}{8}(t^3 - 8), \cos(\frac{\pi t}{4})\right\rangle$, $0 \leqslant t \leqslant 2$? Justify your answer.

5. Let C be a closed curve with positive orientation. For which of the following vector fields could you guarantee
$$\int_C \vec{F}\cdot d\vec{r} = 0?$$
   Explain your answer.
   (a) $\vec{F}(x,y) = \langle y^2 - 2x, 2xy\rangle$
   (b) $\vec{F}(x,y) = \langle 1, 3\rangle$
   (c) $\vec{F}(x,y) = \langle ye^x, e^x + e^y\rangle$
   (d) $\vec{F}(x,y) = \langle xy + y^2, x^2 + 2xy\rangle$
   (e) $\vec{F}(x,y) = \left\langle xe^{\sin y}, xe^{\sin y}\cos y\right\rangle$