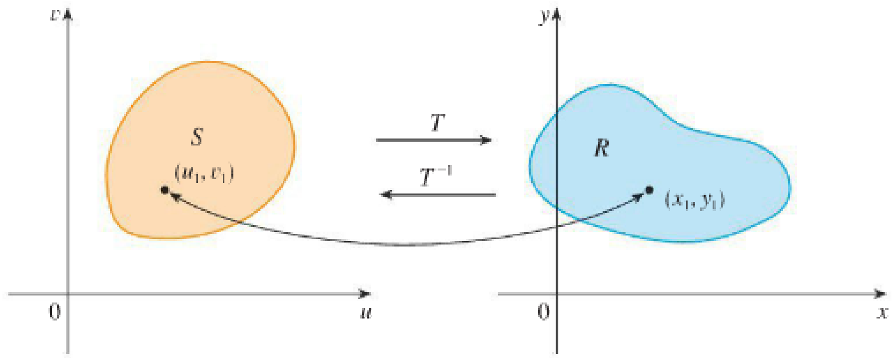
Change of Variables in Multiple Integrals

In Calc 1, a useful technique to evaluate many difficult integrals is by using a u-substitution, which is essentially a change of variable to simplify the integral. Sometimes changing variables can make a huge difference in evaluating a double integral as well, as we have seen already with polar coordinates. Later, we will see the use of changes of variables with triple integrals as well.

In general, say that we have a transformation $T(u,v) = (x,y)$ that maps a region S to a region R (see picture below). All images are taken from Stewart, 8th Edition.



We define the **Jacobian** of the transformation T given by $x = g(u,v)$ and $y = h(u,v)$ as

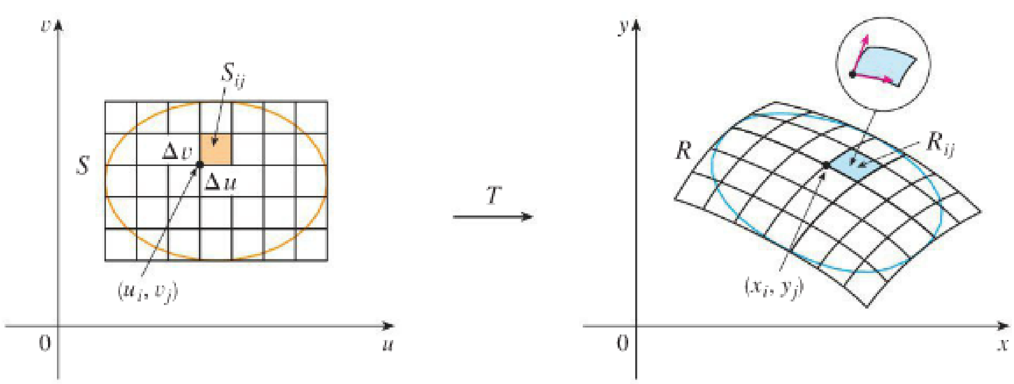
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

We can use this notation to approximate the area ΔA of the region R, the image of T.

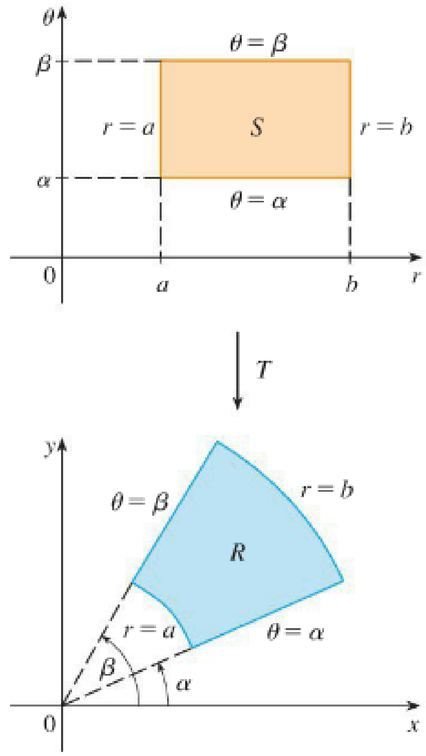
$$\Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v$$

Dividing the region S in the uv-plane into rectangles S_{ij} and calling their images in the xy-plane R_{ij} (see picture below), we can approximate the double integral of a function $f(x,y)$. Taking limits of the double sum, we get the following:

$$\iint_R f(x,y)\, dA = \iint_S f(g(u,v), h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\, du\, dv$$



We have seen one example so far with polar coordinates. In that case, the transformation $T(r, \theta) = (x, y)$ is given by $x = g(r, \theta) = r\cos\theta$ and $y = h(r, \theta) = r\sin\theta$.



The Jacobian of the transformation T is given by

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Therefore, we have that

$$\iint_R f(x, y)\, dx\, dy = \iint_S f(r\cos\theta, r\sin\theta) \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| \, dr\, d\theta = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\; r\, dr\, d\theta.$$

One way to understand the extra factor of r in the integral is to think about how the area of each region is affected if we change the bounds. If we keep the bounds on θ the same, say $\alpha \leqslant \theta \leqslant \beta$, but change the radius from $1 \leqslant r \leqslant 2$ to $101 \leqslant r \leqslant 102$, the area of the region in terms of x and y dramatically increases, even though the area of the rectangle in r and θ would be the same. In short, the bigger the radius, the bigger the area, so the area is scaled up accordingly.

The Jacobian is defined in a similar manner for a transformation with three variables, say $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$. Then we have

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

In particular, we will soon see cylindrical coordinates and spherical coordinates, two examples of three-variable transformations, which have Jacobians of r and $\rho^2 \sin\phi$, respectively.

One of the most useful applications of a change of variables is simplifying otherwise complicated, tedious double or triple integrals. One way to do this is to look at the boundary curves of the region R and see where they are taken under the transformation T. Looking at the boundary of R allows us to determine the region S and use the Jacobian to compute the integral in a different way.

Example 1: Use the transformation given by $x = 2u + v$, $y = u + 2v$ to compute the double integral $\iint\limits_R (x - 3y)\, dA$, where R is the triangular region with vertices $(0, 0)$, $(2, 1)$, and $(1, 2)$.

Example 2: Use the transformation given by $x = 2u$, $y = 3v$ to compute the double integral $\iint\limits_R x^2 \, dA$, where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$.

Exercises (to be completed and turned in at the start of next discussion)

1. Find the Jacobian for each transformation.
   (a) $x = 5u - v$, $y = u + 3v$
   (b) $x = uv$, $y = u/v$
   (c) $x = e^{-r} \sin\theta$, $y = e^{r} \cos\theta$
2. Find the image of the set S under the given transformation.
   (a) S is the square bounded by the lines $u = 0$, $u = 3$, $v = 0$, $v = 3$; $x = 2u + 3v$, $y = u - v$
   (b) S is the triangular region with vertices $(0, 0)$, $(1, 1)$, $(0, 1)$; $x = u^2$, $y = v$
3. Use the transformation given by $x = \frac{1}{4}(u+v)$, $y = \frac{1}{4}(v-3u)$ to compute the double integral $\iint\limits_R (4x + 8y) \, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$.