Name: ______________________________

Score: ________ /20

Applications of Vector Functions

Please staple your work and use this page as a cover page.

1. Show that $\dfrac{d}{dt}[\vec{r}(t) \times \vec{r}\,'(t)] = \vec{r}(t) \times \vec{r}\,''(t)$.

2. If a particle with mass m moves with position vector $\vec{r}(t)$, then its angular momentum is defined as $\vec{L}(t) = m\vec{r}(t) \times \vec{v}(t)$ and its torque as $\vec{\tau}(t) = m\vec{r}(t) \times \vec{a}(t)$. Show that $\vec{L}\,'(t) = \vec{\tau}(t)$. Therefore, if $\vec{\tau}(t) = \vec{0}$ for all t, what can you can conclude about $\vec{L}(t)$? This result is called the Law of Conservation of Angular Momentum.

3. Torque can be rewritten as $\vec{\tau}(t) = m\vec{r}(t) \times \vec{a}(t) = \vec{r}(t) \times \vec{F}(t)$. by noting that $m\vec{a}(t) = \vec{F}(t)$ is the net force. An electron travels through an electric field along a path given by $\vec{r}(t) = \langle t, t^2, t^3 \rangle$. If a force given by $\vec{F}(t) = \langle -2t, t, 4t \rangle$ acts on the particle at time t, find the torque, $\vec{\tau}(t)$.

4. Use a parameterization and an arc length integral to prove that the circumference of a circle is $2\pi R$, where R is the radius.

5. Show that the arc length of $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \leqslant t \leqslant a$, is proportional to a, i.e., the arc length is given by ka for some constant k.

6. A particle P moves with constant angular speed ω around a circle whose center is at the origin and whose radius is R. The particle is said to be in uniform circular motion. Assume that the motion is counterclockwise and that the particle is at the point $(R, 0)$ when $t = 0$. The position vector at time $t \geqslant 0$ is given by $\vec{r}(t) = \langle R\cos(\omega t), R\sin(\omega t) \rangle$.
   (a) Find the velocity vector $\vec{v}(t)$ and show that $\vec{v} \cdot \vec{r} = 0$. Explain why $\vec{v}$ is tangent to the circle and points in the direction of motion.
   (b) Show that $|\vec{v}| = \omega R$, that is that the speed $|\vec{v}|$ is constant and equal to ωR. The period T of the particle is the time required for one complete revolution. Show that
   $$T = \frac{2\pi R}{|\vec{v}|} = \frac{2\pi}{\omega}.$$
   (c) Find the acceleration vector $\vec{a}(t)$. Show that it is proportional to $\vec{r}$ and that it points toward the origin. An acceleration with this property is called a centripetal acceleration. Show that the magnitude of the acceleration vector is $|\vec{a}| = R\omega^2$.
   (d) Suppose that the particle has mass m. Show that the magnitude of the force $\vec{F}$ that is required to produce this motion, called a centripetal force, is
   $$|\vec{F}| = \frac{m|\vec{v}|^2}{R}.$$
   Hint: Since $\vec{F} = m\vec{a}$, then $|\vec{F}| = m|\vec{a}|$.

7. For a given vector function $\vec{r}(t)$, if $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$ for all t, show that $\vec{r}(t)$ is contained in a sphere.

   Hint: If $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$, then $\vec{r}(t) \cdot \vec{r}\,'(t) + \vec{r}\,'(t) \cdot \vec{r}(t) = 0$ as well. Now integrate both sides.