

Finding Absolute Maxima and Minima

In Calculus I, we first learned how to find and classify critical points, which allow us to find the location of local maxima and minima. We also discussed the application of finding the absolute maximum and minimum values of a function $y = f(x)$ over a closed interval $[a, b]$. Recall that the absolute maximum and minimum can only occur at either critical points or at the endpoints of the interval but nowhere else, assuming that $f(x)$ is continuous on $[a, b]$. This is known as the Extreme Value Theorem, and our goal is to now extend these ideas to functions of two or more variables.

Once again, it should be clear that an absolute maximum or minimum value can occur at a critical point, but what corresponds to the endpoints? First, absolute extrema are only guaranteed to exist over a set D that is both *closed* and *bounded*. What does that mean? A set D is called closed if it includes its boundary. For example, the set defined by $x^2 + y^2 \leq 1$ is closed, but $x^2 + y^2 < 1$ is not (the latter set would have a dashed circle while the first would be solid). A set D is called bounded if it doesn't extend infinitely in any direction. Mathematically, this is equivalent to being able to fit the set D inside of a disk with some finite radius.

We can now state the Extreme Value Theorem for a function of two variables. If $f(x, y)$ is continuous on a closed, bounded set D in the plane, then f will attain an absolute maximum and minimum value at either a critical point inside D or on the boundary of the set D .

Example 1: Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 + x^2y + 4$ on the set D , the rectangle given by $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. The critical points of $f(x, y)$ are $(0, 0)$, $(\sqrt{2}, -1)$, and $(-\sqrt{2}, -1)$.

Sometimes the set D can be more easily described by one equation, which makes the process of checking the boundary more straightforward. However, the computations can become more tedious.

Example 2: Find the absolute maximum and minimum values of $f(x, y) = xy^2$ on the set D given by $x^2 + y^2 \leq 3$. The critical points of $f(x, y)$ are all points on the x -axis, namely points of the form $(x, 0)$.

In the event that the boundary curve of a set D is given by one equation, we can use another method to find the absolute maximum and minimum values. This curve can be thought of as a level curve of some function $g(x, y)$, so we write the curve as $g(x, y) = k$ (some constant k). This is called a **constraint**, and solving many real-world problems goes hand-in-hand with solving constrained optimization problems.

The method that we will use is called the method of **Lagrange multipliers**, which essentially relies on the observation that a function $f(x, y)$ is maximized or minimized over a curve $g(x, y) = k$ whenever the functions' gradient vectors are parallel at a point on the curve. This works for functions defined with any number of variables, but we will state the system in the case of two variables. Namely, given a function $f(x, y)$ that we wish to maximize or minimize and a constraint $g(x, y) = k$, we seek solutions to the system of equations

$$\begin{aligned}\vec{\nabla}f(x, y) &= \lambda \cdot \vec{\nabla}g(x, y) \\ g(x, y) &= k\end{aligned}$$

Here, λ is a constant, called a Lagrange multiplier. Also, note that $\vec{\nabla}f = \lambda\vec{\nabla}g$ is really *two* equations: $f_x = \lambda g_x$ and $f_y = \lambda g_y$, which come from the corresponding components of the gradient vectors.

Example 3: Find the maximum and minimum values of $f(x, y) = 3x + y$ subject to the constraint $x^2 + y^2 = 10$.

Sometimes we only obtain one value from this method, meaning that there is either a maximum with no minimum or vice versa. It is then important to check analytically to determine which one has been found.

Example 4: Find the maximum and minimum values of $f(x, y) = x^2 + y^2$ subject to the constraint $xy = 1$.

Exercises (to be completed and turned in at the start of next discussion)

1. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + y^2 - 2x$ over the triangular region D with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$.
2. Find the absolute maximum and minimum values of the function $f(x, y) = y^2 - x^2$ over the region given by $x^2 + 4y^2 \leq 4$. (Hint: use Lagrange multipliers to find the max and min on the boundary)
3. Find the maximum area of a rectangle with sides measuring x and y if the perimeter is 14. Is there a minimum value of the area?
4. Find the maximum and minimum values of the function $f(x, y) = x^2 + y^2 + z^2$ subject to the constraint $x^4 + y^4 + z^4 = 1$. (Remember that you want to solve $\vec{\nabla} f = \lambda \vec{\nabla} g$ and $g(x, y) = k$)