MATH 1071
FINAL EXAM REVIEW

Q1 a) $\int 5\,dx = 5x + C$

b) $\int x^{99}\,dx = \frac{1}{100}x^{100} + C$

c) $\int x^{-99}\,dx = -\frac{1}{98}x^{-98} + C$

d) $\int \frac{5}{y^3}\,dy = 5\int y^{-3}\,dy = 5\left(-\frac{1}{2}\right)x^{-2} + C = -\frac{5}{2}\frac{1}{x^2} + C$

e) $\int \frac{y^{3/2}}{\sqrt{2}}\,dy = \frac{1}{\sqrt{2}}\int y^{3/2}\,dy = \frac{2}{5\sqrt{2}}y^{5/2} + C$

f) $\int \sqrt[3]{u^2}\,du = \int u^{2/3}\,du = \frac{3}{5}u^{5/3} + C$

g) $\int (6x^2 + 4x)\,dx = \int 6x^2\,dx + \int 4x\,dx$

$= 6\int x^2\,dx + 4\int x\,dx = \frac{6}{3}x^3 + \frac{4}{2}x^2 + C$

$= 2x^3 + 2x^2 + C$

h) $\int\left(\frac{3}{t^2} - 6t^2\right)dt = -\frac{3}{t} - 2t^3 + C$

i) $\int\left(x + \frac{1}{x}\right)dx = \frac{x^2}{2} + \ln|x| + C$

j) $\int\left(\pi + \frac{1}{x}\right)dx = \pi x + \ln|x| + C$

k) $\int \frac{t+1}{\sqrt{t}}\,dt = \int \left(\frac{t}{\sqrt{t}} + \frac{1}{\sqrt{t}}\right) dt$

$= \int \sqrt{t}\,dt + \int \frac{1}{\sqrt{t}}\,dt = \int t^{1/2}\,dt + \int t^{-1/2}\,dt$

$= \frac{2}{3} t^{3/2} + 2t^{1/2} + C$

l) $\int (e^x - 1/x)\,dx = \int e^x\,dx - \int x^{-1}\,dx$

$= e^x - \ln|x| + C.$

<u>Q2</u> a) $\int 6(3x+1)^{10}\,dx$

let $u = 3x+1$ so $du = 3\,dx$ and $2du = 6dx$.

Then $\int 6(3x+1)^{10}\,dx = \int 2u^{10}\,du = \frac{2}{11}u^{11} + C$

$= \frac{2}{11}(3x+10)^{11} + C$

b) $\int x(3-x^2)^7\,dx$

let $u = 3-x^2$ so $du = -2x\,dx$ and $-\frac{1}{2}du = x\,dx$.

Then $\int x(3-x^2)^7 dx = \int -\frac{1}{2}u^7\,du = -\frac{1}{16}u^8 + C$

$= -\frac{1}{16}(3-x^2)^8 + C$

c) $\int (x^3+2)(x^4+8x+3)^{1/3}\,dx$

let $u = x^4+8x+3$ so $du = (4x^3+8)dx$ and $\frac{1}{4}du = (x^3+2)dx$

Then $\int (x^3+2)(x^4+8x+3)^{1/3} = \frac{1}{4}\int u^{1/3}\,du = \frac{3}{16}u^{4/3} + C$

$= \frac{3}{16}(x^4+8x+3)^{4/3} + C$

d) $\int 2\sqrt{x+1}\, dx$

let $u = x+1$ so $du = dx$ and $2du = 2dx$

Then $\int 2\sqrt{x+1}\, dx = 2\int u^{1/2}\, du = \frac{4}{3} u^{3/2} + C$

$= \frac{4}{3}(x+1)^{3/2} + C.$

e) $\int \sqrt[3]{x+1}\, dx$

let $u = x+1$ so $du = dx.$

Then $\int \sqrt[3]{x+1}\, dx = \int u^{1/3}\, du = \frac{3}{4} u^{4/3} + C$

$= \frac{3}{4}(x+1)^{4/3} + C$

f) $\int \frac{\ln 2x}{x}\, dx$

let $u = \ln 2x$ so $du = 1/x\, dx$

Then $\int \frac{\ln 2x}{dx} dx = \int u\, du = \frac{u^2}{2} + C = \frac{(\ln 2x)^2}{2} + C$

g) $\int \frac{1}{2x+1}\, dx$

let $u = 2x+1$ so $du = 2dx$ and $\frac{1}{2} du = dx.$

Then $\int \frac{1}{2x+1}\, dx = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|2x+1| + C.$

h) $\int \frac{e^{-x}}{e^{-x}+1}\,dx$

let $u = e^{-x}+1$ so $du = -e^{-x}dx$ and $-du = e^{-x}dx$.

Then $\int \frac{e^{-x}}{e^{-x}+1}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C$

$= -\ln(e^{-x}+1) + C$

i) $\int \frac{1}{x\ln x}\,dx$

let $u = \ln x$ so $du = 1/x\,dx$.

Then $\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C$

$= \ln|\ln x| + C$

j) $\int \frac{1}{x\ln x^2}\,dx$

let $u = \ln x^2$ so $du = 2/x\,dx$ and $1/2\,du = 1/x\,dx$.

Then $\int \frac{1}{x\ln x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|\ln x^2| + C.$

Q3

a) $\int_1^2 4x^3\,dx = x^4\Big|_1^2 = 2^4 - 1^4 = 15$

b) $\int_{-2}^2 3x^4\,dx = \frac{3}{5}x^5\Big|_{-2}^2 = \frac{3}{5}\cancel{(}2)^5 - \frac{3}{5}(-2)^5$

$= \frac{3}{5}(32) - \frac{3}{5}(-32) = \frac{192}{5}$

c) $\int_{-1}^0 (9x^2-1)\,dx = (3x^3 - x)\Big|_{-1}^0 = 0 - \left(3(-1)^3 - (-1)\right)$

$= -(-3+1) = 2$

d) $\int_1^2 (x^{-2} + 3x^{-4})\,dx = \left(-\frac{1}{x} - \frac{1}{x^3}\right)\Big|_1^2$

$= \left(-\frac{1}{2} - \frac{1}{2^3}\right) - \left(-\frac{1}{1} - \frac{1}{1^3}\right) = \frac{11}{8}$

e) $\int_{-1}^0 e^{-x}\,dx = \left(-e^{-x}\right)\Big|_{-1}^0 = -e^0 + e^1 = -1 + e.$

f) $\int_{-2}^{-1} e^{2x}\,dx = \left(\frac{1}{2}e^{2x}\right)\Big|_{-2}^{-1} = \frac{1}{2}e^{-2} - \frac{1}{2}e^{-4}$

g) $\int_2^4 \frac{3}{x}\,dx = \left(3\ln|x|\right)\Big|_2^4 = 3\ln 4 - 3\ln 2$

h) $\int_{-1}^0 (1+2x)^5\,dx = \frac{1}{2}\left((1+2x)^6\right)\Big|_{-1}^0 = \frac{1}{2}(1+0)^6 - \frac{1}{2}(1-2)^6$

$= \frac{1}{2} + \frac{1}{2} = 0$

i) $\int_{-1}^1 x e^{x^2+1}\,dx = \left(\frac{1}{2}e^{x^2+1}\right)\Big|_{-1}^1 = \frac{1}{2}e^2 - \frac{1}{2}e^2 = 0$

j) $\int_1^2 \frac{1}{2x+1}\,dx = \left(\frac{1}{2}\ln|2x+1|\right)\Big|_1^2 = \frac{1}{2}\ln 5 - \frac{1}{2}\ln 3$

k) $\displaystyle\int_{-2}^{0} \frac{x}{x^2+1}\,dx = \left(\frac{1}{2}\ln(x^2+1)\right)\Big|_{-2}^{0}$

$= \frac{1}{2}\ln 1 - \frac{1}{2}\ln 5 = -\frac{1}{2}\ln 5.$

Q4 a) $\bar{f}(x) = \displaystyle\frac{1}{b-a}\int_a^b f(x)\,dx$

$= \displaystyle\frac{1}{10}\int_0^{10} x\,dx = \frac{1}{10}\left(\frac{x^2}{2}\right)\Big|_0^{10} = \frac{1}{10}\frac{10^2}{2} - \frac{1}{10}\frac{0^2}{2}$

$= 10/2 = 5.$

b) $\bar{f} = \displaystyle\frac{1}{2}\int_{-1}^{1} x^3\,dx = \frac{1}{2}\left(\frac{x^4}{4}\right)\Big|_{-1}^{1} = \frac{1}{2}\cdot\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{4} = 0$

c) $\bar{f} = \displaystyle\frac{1}{\ln 2}\int_0^{\ln 2} e^x\,dx = \frac{1}{\ln 2}\left(e^x\right)\Big|_0^{\ln 2}$

$= \displaystyle\frac{1}{\ln 2}e^{\ln 2} - \frac{1}{\ln 2}e^0 = \frac{2}{\ln 2} - \frac{1}{\ln 2} = \frac{1}{\ln 2}.$

d) $\bar{f} = \displaystyle\frac{1}{2}\int_0^2 ex(x-1)\,dx$

$= \displaystyle\frac{1}{2}\int_0^2 ex^2 - ex\,dx$

$= \displaystyle\frac{1}{2}\left(\frac{ex^3}{3} - \frac{ex^2}{2}\right)\Big|_0^2$

$= \displaystyle\frac{1}{2}\left(\frac{8e}{3} - \frac{4e}{2} - 0\right) = \frac{e}{3}.$

Q5 a) $A = \int_1^2 x^3\,dx = \frac{x^4}{4}\Big|_1^2 = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}.$

b) $A = \int_{-1}^{0} 3 - e^{-x}\,dx = \left(3x + e^{-x}\right)\Big|_{-1}^{0}$

$= 3(0) + e^0 - 3(-1) - e^1 = +1 + 3 - e = 4 - e$

c) $A = \int_0^1 x^{1/2} - x\,dx = \left(\frac{2}{3}x^{3/2} - \frac{x^2}{2}\right)\Big|_0^1 = \frac{1}{6}$

d) $x^2 - 2x + 1 = x + 1$ when $x^2 - 3x = 0$
i.e. when $x(x-3) = 0$ so when $x = 0$ and $x = 3$

$A = \int_0^3 x + 1 - (x^2 - 2x + 1)\,dx$
$= \int_0^3 -x^2 + 3x\,dx = \frac{9}{2}$

e) $x^2 = 8 - x^2$ when $2x^2 = 8$ i.e. when $x^2 = 4$
so when $x = \pm 2$

$A = \int_{-2}^{2} 8 - x^2 - x^2\,dx = \int_{-2}^{2} 8 - 2x^2\,dx = \frac{64}{3}$

f) $e^{2x} = 3^{-2x}$ when $2x = \ln(3^{-2x})$
i.e. when $2x = (-2x)\ln(3)$ so $x = 0.$

$A = \int_{-1}^{0} 3^{-2x} - e^{2x}\,dx + \int_0^2 e^{2x} - 3^{-2x}\,dx$

$= \left(-\frac{1}{2}3^{-2x} \cdot \frac{1}{\ln 3} - \frac{1}{2}e^{2x}\right)\Big|_{-1}^{0} + \left(\frac{1}{2}e^{2x} + \frac{1}{2}3^{-2x} \cdot \frac{1}{\ln 3}\right)\Big|_0^2$

$= \left(-\frac{1}{2\ln 3} - \frac{1}{2} + \frac{1}{2\ln 3}3^2 + \frac{1}{2}e^{-2}\right) + \left(\frac{1}{2}e^4 + \frac{1}{2\ln 3}3^{-4} - \frac{1}{2} - \frac{1}{2\ln 3}\right)$

g) $x^3 - 3x = 2x^2$ when $x^3 - 2x^2 - 3x = 0$

i.e. when $x(x+1)(x-3) = 0$

i.e. when $x = 0$, $x = -1$ or $x = 3$.

So $$A = \int_{-1}^{0} x^3 - 3x - 2x^2 \, dx + \int_{1}^{3} 2x^2 - x^3 + 3x \, dx$$

$$= \left(\frac{x^4}{4} - \frac{3x^2}{2} - \frac{2x^3}{3} \right) \Big|_{-1}^{0} + \left(\frac{2x^3}{3} - \frac{x^4}{4} + \frac{3x^2}{2} \right) \Big|_{1}^{3}$$

$$= \frac{7}{12} + \frac{28}{3} = \frac{2164}{3}.$$

h) $y = \sqrt[3]{x}$, $y = x$, $x = -8$, $x = 1$

<u>Note:</u> when $x < 0$ we take $\sqrt[3]{x}$ to denote the real-valued cube root of x

$$A = \int_{-8}^{-1} \sqrt[3]{x} - x \, dx + \int_{-1}^{0} x - \sqrt[3]{x} \, dx + \int_{0}^{1} \sqrt[3]{x} - x \, dx$$

$$= \left(\frac{3}{4} x^{4/3} - \frac{x^2}{2} \right) \Big|_{-8}^{-1} + \left(\frac{x^2}{2} - \frac{3}{4} x^{4/3} \right) \Big|_{-1}^{0} + \left(\frac{3}{4} x^{4/3} - \frac{x^2}{2} \right) \Big|_{0}^{1}$$

$$= \left(\frac{3(-1)^{4/3}}{4} - \frac{(-1)^2}{2} - \frac{3}{4}(-8)^{4/3} + \frac{(-8)^2}{2} \right) + \left(0 - \frac{(-1)^2}{2} + \frac{3}{4}(-1)^{4/3} \right) + \left(\frac{3(1)^{4/3}}{4} - \frac{(1)^2}{2} - 0 \right)$$

$$= \left(\frac{3}{4} - \frac{1}{2} - \frac{3}{4}(16) + \frac{64}{2} \right) + \left(-\frac{1}{2} + \frac{3}{4} \right) + \left(\frac{3}{4} - \frac{1}{2} \right)$$

$$= \frac{85}{4}$$