University of Connecticut
Department of Mathematics

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Math 1071Q

Exam 2, Fall 2015

Duration: 120 minutes

Name: ______________________ Section: ______ Instructor's Name: ______________________

Page	Points	Score
2	12	
3	16	
4	10	
5	20	
6	16	
7	14	
8	12	
Total:	100	

1. Write clearly. Points may be deducted if your work is messy or your answer unclear;
2. Answer the questions in the space provided. You may use the back of the page if necessary;
3. You must show your work or explain your solution, otherwise points may be deducted;
4. No credit will be given for incorrect steps nor will credit be given for correct solutions arrived at by incorrect means;
5. If you make an unnecessary approximation in your solution to a problem, your answer will be judged on its accuracy. Points may be deducted for poor or inappropriate approximation;

1. (a) (6 points) Find the derivative of $f(x) = \ln(3x^2 + 3x - 1)$.

Solution: Let $y = \ln(3x^2 + 3x - 1)$ and let $u = 3x^2 + 3x - 1$. Then

$$\frac{du}{dx} = 6x + 3$$

and $y = \ln u$.
So,

$$f'(x) = \frac{dy}{dx} = \frac{dy}{dy} \cdot \frac{du}{dx} = \frac{1}{u} \cdot (6x + 3) = \frac{6x + 3}{3x^2 + 3x - 1}.$$

(b) (6 points) Find the derivative of $f(x) = e^{\sqrt{x}} + \sqrt{\ln x}$

Solution:

$$f'(x) = \frac{1}{2} \cdot \frac{1}{\sqrt{x}} e^{\sqrt{x}} + \frac{1}{2} \cdot \frac{1}{x} \cdot \frac{1}{\sqrt{\ln x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}} + \frac{1}{2x\sqrt{\ln x}}.$$

2. (a) (8 points) Find where the function

$$f(x) = x - \ln(x)$$

is increasing and decreasing. What are the critical points of $f(x)$?

Solution: The domain of $f(x)$ is $(0, \infty)$.
The derivative of $f(x)$ is $f'(x) = 1 - \frac{1}{x}$. So $f'(x) = 0$ when $x = 1$ (and is undefined when $x = 0$).
Therefore $(1, f(1)) = (1, 1)$ is the only critical point of f.
Moreover, $f'(1/2) = -1 < 0$ so f is decreasing on $(0, 1)$ and $f'(2) = 1/2 > 0$ so f is increasing on $(1, \infty)$.

(b) (8 points) Find where the function

$$f(x) = x^4$$

is concave up and concave down. What are the inflection points of $f(x)$?

Solution: The first derivative is $f'(x) = 4x^3$ and the second derivative is $f''(x) = 12x^2$.
So $f''(x)$ exists everywhere and $f''(x) = 0$ when $x = 0$.
Moreover, $f''(-1) = 12 > 0$ and $f''(1) = 12 > 0$. So, f is concave up on $(-\infty, 0)$ and $(0, \infty)$.
Therefore, f does not have a point of inflection.

3. (a) (5 points) Compute the limit

$$\lim_{x\to\infty} \frac{x^2+5x+6}{2x^2-6x}.$$

Solution:

$$\lim_{x\to\infty} \frac{x^2+5x+6}{2x^2-6x} = \lim_{x\to\infty} \frac{1+\frac{5}{x}+\frac{6}{x^2}}{2-\frac{6}{x}} = \frac{1}{2}.$$

(b) (2 points) Compute the limit

$$\lim_{x\to 2} \frac{x^2+5x+6}{2x^2-6x}.$$

Solution:

$$\lim_{x\to 2} \frac{x^2+5x+6}{2x^2-6x} = \frac{20}{-4} = -5.$$

(c) (3 points) What do the answers to parts (a) and (b) tell us about the asymptotes of the graph of

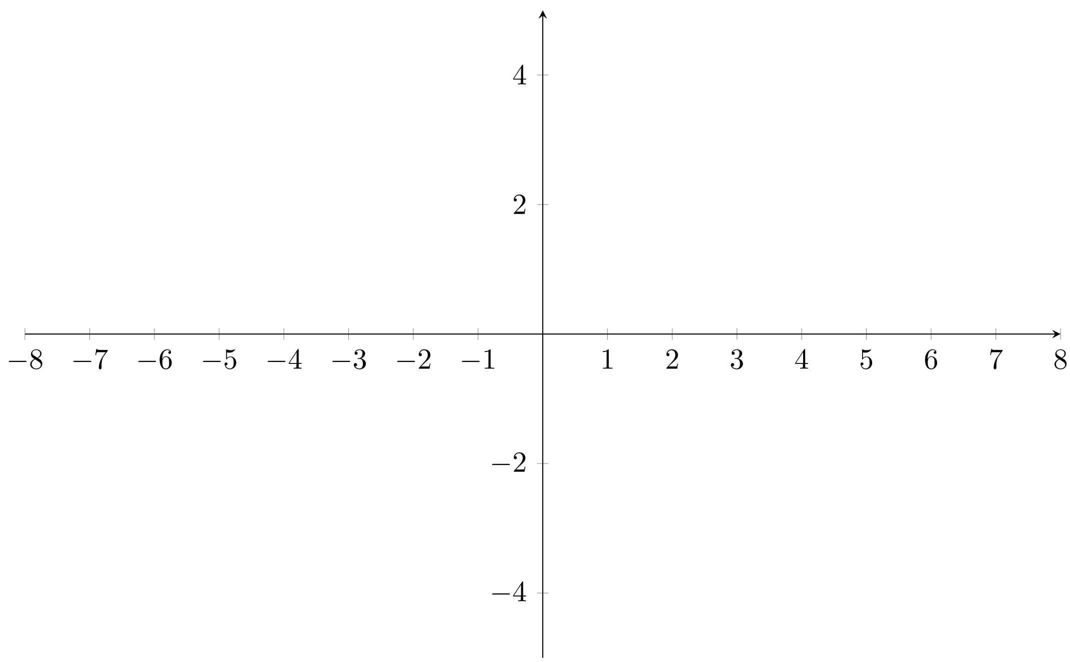
$$f(x) = \frac{x^2+5x+6}{2x^2-6x}.$$

Solution: Part (a) tells us that the line $y = \frac{1}{2}$ is a horizontal asymptote to the graph of $f(x)$.
Part (b) tells us nothing about the asymptotes of the graph of $f(x)$.

4. (12 points) Sketch the graph of a function $f(x)$ that has all of the following properties:

- $f(-7) = f(-4) = f(1) = f(6) = 0,\ f(-2) = 4,\ f(4) = -2$
- $f'(-2) = f'(4) = 0$
- $f''(0) = 0$
- $\lim_{x\to-\infty} f(x) = -2$, $\lim_{x\to\infty} f(x) = 2$
- $\lim_{x\to-5^-} f(x) = \infty$, $\lim_{x\to-5^+} f(x) = -\infty$
- $f'(x) > 0$ on $(-\infty,-5)\cup(-5,-2)\cup(4,\infty)$, $f'(x) < 0$ on $(-2,4)$
- $f''(x) > 0$ on $(-\infty,-5)\cup(0,6)$, $f''(x) < 0$ on $(-5,0)\cup(6,\infty)$.

You should clearly label any points of inflection on the graph.



5. Suppose the demand function for a certain commodity is $x = (10-2p)^2$.
   (a) (5 points) Find the elasticity of demand $E(p)$.
   (b) (3 points) Find the price where the revenue is optimised.

Solution:

$$\frac{dp}{dx} = 2(10-2p)(-2) = -4(10-2p)$$

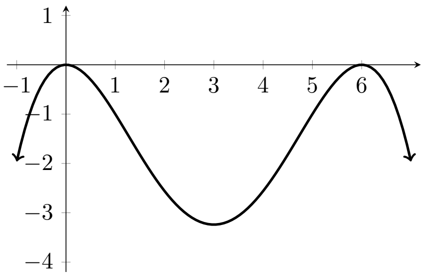
so

$$E(p) = -\frac{p}{(10-2p)^2}(-4)(10-2p) = \frac{4p}{10-2p}.$$

The revenue is optimised when $E = 1$, i.e. when

$$\frac{4p}{10-2p} = 1 \text{ that is } p = \frac{10}{6} = \frac{5}{3}.$$

6. (a) (8 points) The following is a graph of $f(x)$.



i. What are the critical values of $f(x)$?

Solution: $x = 0, 3$ and 6.

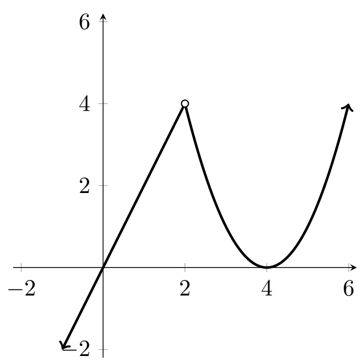
ii. Determine the intervals on which $f(x)$ is increasing and decreasing.

Solution: f is increasing on $(-\infty, 0) \cup (3, 6)$ and decreasing on $(0, 3) \cup (6, \infty)$.

iii. Find the x-coordinates for the relative maxima of $f(x)$.

Solution: $x = 0$ and $x = 6$.

(b) (8 points) The following is a graph of $f'(x)$. You may assume that the domain of $f(x)$ is $(-\infty, \infty)$.



i. What are the critical values of $f(x)$?

Solution: $x = 0, 2$ and 4.

ii. Determine the intervals on which $f(x)$ is increasing and decreasing.

Solution: f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

iii. Determine the intervals where $f(x)$ is concave up and concave down.

Solution: f is concave up on $(-\infty, 2) \cup (4, \infty)$ and concave down on $(2, 4)$.

iv. Find the x-coordinates of any relative extrema of $f(x)$.

Solution: There is a relative minimum at $x = 0$.

7. (a) (6 points) Find the absolute extreme values of the function

$$f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x$$

on the interval $[0, 4]$.

Solution:

$$f'(x) = 3x^2 - 5x + 6 = (x-2)(x-3)$$

So, $f'(x)$ exists everywhere and $f'(x) = 0$ when $x = 2$ or $x = 3$.
Now, $f(0) = 0$, $f(2) = 14/3$, $f(3) = 9/2$ and $f(4) = 16/3$.
Therefore, the absolute minimum value of $f(x)$ on $[0, 4]$ is 0 and the absolute maximum value is $16/3$.

(b) (8 points) Find the absolute extreme values (if they exist) of the function

$$f(x) = 3x + \frac{1}{x^3}$$

on the interval $(0, \infty)$

Solution:

$$f'(x) = 3 - \frac{3}{x^4} \text{ and } f''(x) = \frac{12}{x^5}.$$

So $f'(x) = 0$ when $x = \pm 1$ and is undefined when $x = 0$.
Moreover, $f''(1) = 12 > 0$. So $f(1) = 4$ is the absolute minimum value on $(0, \infty)$.
Finally, $\lim_{x \to 0^+} f(x) = \infty$ so there is no absolute maximum value on $(0, \infty)$.

8. (12 points) If 300 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

Solution: Assume that x is the side-length of the base of the box and y is the height of the box (both in cm).

Then the surface-area A is given by

$$A = x^2 + 4xy = 300$$

and so

$$y = \frac{300 - x^2}{4x}.$$

The volume V is given by

$$V = x^2 y$$

and so

$$V = x^2 \left(\frac{300 - x^2}{4x} \right) = 75x - \frac{x^3}{4}.$$

We wish to maximise V as a function of x where $x \in (0, \sqrt{300})$.

Now,

$$V'(x) = 75 - \frac{3x^2}{4}$$

and so the critical values of V are $x = \pm 10$ but only $x = 10$ is in the domain of V.

Moreover, $V'(x) > 0$ for $x < 10$ and $V'(x) < 0$ for $x > 10$, so $x = 10$ is a relative maximum for V.

Finally, since $x = 10$ is the only critical value of V in the open interval $(0, \sqrt{300})$, the absolute maximum volume of $V(10) = 500$ cm^3 is attained when $x = 10$.