i. Use a linear approximation to approximate $\sqrt{25.5}$.

Solution. We know that $f(x) \approx f(c) + f'(c)(x-c)$ is the linnear approximation of a function $f(x)$ at the point $c$. We need to guess a function $f(x)$ and a point $c$ close to $x$. The natural guesses are

$$f(x) = \sqrt{x} \quad \text{and} \quad c = 25$$

since $\sqrt{25} = 5$. Taking the derivative of $f(x)$, we get

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(c) = f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10}.$$

Since $f(c) = 5$, we have

$$\sqrt{x} = f(x) \approx 5 + \frac{1}{10}(x-25).$$

Plugging $x = 25.5$, we get

$$\sqrt{25.5} \approx 5 + \frac{1}{10}(25.5-25) = 5 + \frac{1}{10}(0.5) = 5.05$$

ii. Use a linear approximation to approximate $(1.8)^3$.

Solution. We let $f(x) = x^3$ and $c = 2$. Then

$$f(c) = 2^3 = 8, \quad f'(x) = 3 \cdot x^2 \Rightarrow f'(c) = 3 \cdot 2^2 = 12$$

Hence, we get

$$x^3 \approx 8 + 12(x-2)$$

Plugging $x = 1.8 \Rightarrow (1.8)^3 \approx 8 + 12(1.8-2) = 8 + 12(-0.2) = 8 - 2.4$
$= 5.6$

Alternatively, without taking derivatives, we can do the following

$$x^3 = [(x-2)+2]^3 = (x-2)^3 + 6(x-2)^2 + \underline{12(x-2) + 8} \quad \star$$

where I'm using $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ formula. For $x$ close to

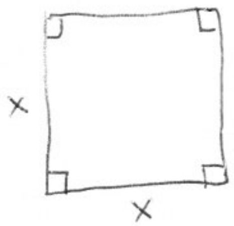
$$x^3 \approx \underline{12(x-2) + 8} \quad \text{(where we take the linear part of } \star\text{)}$$

Plugging in $x = 1.8$, we get $(1.8)^3 \approx 12(1.8-2) + 8 = 5.6$ as above.

Remark. You don't need to know this alternate way for the exam.
Can you guess the best quadratic approximation of $(1.8)^3 \approx$ ?

iii. If the side of a square decreases from 4 inches to 3.8 inches, use the linear approximation formula to estimate the change in area.

Solution. Imagine a square with sides of lenght $x$:




The area of this square is $A(x) = x^2$.
We want to approximate $A(3.8) - A(4) \approx ??$
We know from above that

$$A(x) \approx A(c) + A'(c)(x-c)$$

$$\iff A(x) - A(c) \approx A'(c)(x-c).$$

Therefore, using $c = 4$, we get

$$A'(x) = 2x \Rightarrow A'(c) = 8 \quad \text{and} \quad A(c) = 4^2 = 16$$

Plugging in $x = 3.8$, we get:

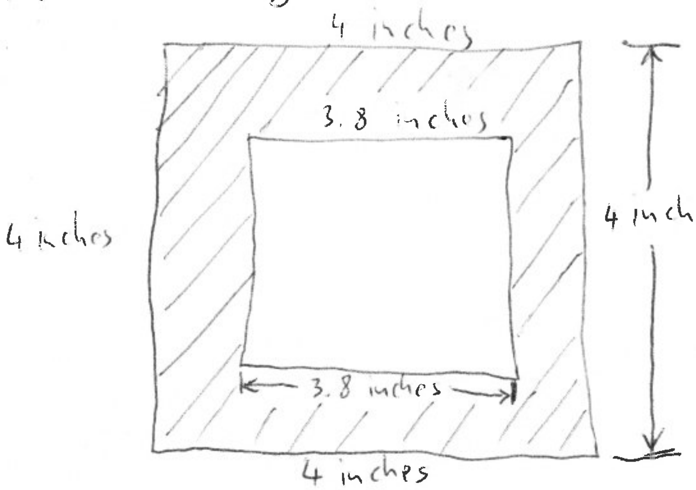
$$A(3.8) - A(4) \approx 8\,(3.8-4) = -1.6$$

Therefore the area of the square decreased by approximately $1.6 \text{ in}^2$. We know it decreased since we got a negative answ

Remark. We know the exact difference in area is

$$A(3.8) - A(4) = 3.8^2 - 4^2 = (3.8-4)(3.8+4) = -0.2 \cdot 7.8 = -1.5$$

which is quite close to our approximation of $-1.6$.
Geometrically we are finding (approximating) the shaded region below:




Shaded area $\approx 1.6$ inches

7 i. If $x = \frac{1}{10+p}$, then find the elasticity E at the points $p=5$, $p=10$ and $p=100$ and determine whether the demand is elastic, inelastic or of unit elasticity.

Solution. Recall that $E = -\frac{p}{x}\frac{dx}{dp}$.

① If $E > 1$, then demand is elastic,

② If $E < 1$, then demand is inelastic.

③ If $E = 1$, then demand has unit elasticity.

Step 1. Find $\frac{dx}{dp}$. From problem, we get

$$\frac{dx}{dp} = \left[(10+p)^{-1}\right]' = -1\cdot(10+p)^{-2} = -\frac{1}{(10+p)^2}.$$

Step 2. Now plugging this into the formula above together with $x = \frac{1}{10+p}$, we get

$$E = -\frac{p}{\frac{1}{10+p}}\cdot\left(-\frac{1}{(10+p)^2}\right) = \frac{-p\cdot(-1)}{\frac{1}{10+p}\cdot(10+p)^2} = \frac{p}{10+p}$$

Step 3.

@ $p=5$: $E = \frac{5}{10+5} = \frac{5}{15} = \frac{1}{3} < 1 \overset{②}{\Rightarrow}$ demand is inelastic.

@ $p=10$: $E = \frac{10}{10+10} = \frac{1}{2} < 1 \overset{②}{\Rightarrow}$ demand is inelastic.

@ $p=100$: $E = \frac{100}{10+100} = \frac{100}{110} < 1 \overset{②}{\Rightarrow}$ demand is inelastic.

Remark. Note that for any $p \geq 0$, we have $p < p+10$. Therefore, $E = \frac{p}{p+10} < 1$ for any $p \geq 0$. As $p \to \infty$, we have $\lim_{p\to\infty} E(p) = 1$. Hence, increasing price will increase revenu[e]. We can never have unit elasticity or elastic dem[and].

7 ii. If $x = \frac{1}{p^3}$, then find the value (if any) of $x$ such that $E = 1$ and the values (if any) of $x$ for which revenue is maximized.

<u>Solution.</u> Note that from class we know that revenue is maximized precisely when $E = 1$ (i.e. when we have unit elasticity). So, the answer to both questions is the same.

<u>Step 1.</u> Find $\frac{dx}{dp}$. From problem, ~~and~~ we have

$$\frac{dx}{dp} = (p^{-3})' = -3p^{-3-1} = -3 \cdot p^{-4} = -3 \cdot \frac{1}{p^4} = -\frac{3}{p^4}$$

<u>Step 2.</u> Plugging this into the formula for $E$, together with $x = \frac{1}{p^3}$, we get:

$$E = -\frac{p}{\frac{1}{p^3}} \cdot \frac{dx}{dp} = -\frac{p}{\frac{1}{p^3}} \cdot \left(-\frac{3}{p^4}\right) = \frac{(-p)(-3)}{\frac{1}{p^3} \cdot p^4} = \frac{3p}{p} = 3$$

<u>Step 3.</u> Now we are supposed to set $E = 1$ and solve for $p$. However, $E = 3$ and so there's no $p$ that will make $E = 1$.

<u>Step 4.</u> Here you are supposed to take the values of $p$ and plug them into $x = \frac{1}{p^3}$, but since <u>Step 3</u> didn't yield any $p$, there's no $x$ for which revenue is maximized.

In conclusion, there's no solution to our problem.