**4 i.** $\lim_{x\to\infty} \frac{x^2}{x^3+1} = \lim_{x\to\infty} \left(\frac{x^2}{x^3+1}\right)\frac{\frac{1}{x^3}}{\frac{1}{x^3}}$

$= \lim_{x\to\infty} \frac{\frac{x^2}{x^3}}{\frac{x^3}{x^3}+\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{1}{x} \to 0}{\underbrace{1}_{\to 1}+\underbrace{\frac{1}{x^3}}_{\to 0}}$

$= \frac{0}{1} = \boxed{0}$

**4 ii.** $\lim_{x\to\infty} \sqrt{x} = \boxed{\infty}$ since $\sqrt{x}$ becomes unbounded as $x$ approaches $\infty$

**4 iii.** $\lim_{x\to\infty} \frac{x^4 - x^2 + x - 1}{x^3+1}$

$= \lim_{x\to\infty} \left(\frac{x^4-x^2+x-1}{x^3+1}\right)\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}}$

$= \lim_{x\to\infty} \frac{\frac{x^4}{x^3} - \frac{x^2}{x^3} + \frac{x}{x^3} - \frac{1}{x^3}}{\frac{x^3}{x^3}+\frac{1}{x^3}}$

$= \lim_{x\to\infty} \frac{\underbrace{x}_{\to\infty} - \underbrace{\frac{1}{x}}_{\to 0} + \underbrace{\frac{1}{x^2}}_{\to 0} - \underbrace{\frac{1}{x^3}}_{\to 0}}{\underbrace{1}_{\to 1} + \underbrace{\frac{1}{x^3}}_{\to 0}} = \boxed{\infty}$

**4 iv.** $\lim_{x\to\infty} \frac{x^2-1}{\sqrt{x}-3} = \lim_{x\to\infty} \frac{x^2-1}{x^{1/2}-3}$

$= \lim_{x\to\infty} \left(\frac{x^2-1}{x^{1/2}-3}\right)\cdot\frac{\frac{1}{x^{1/2}}}{\frac{1}{x^{1/2}}} = \lim_{x\to\infty} \frac{\frac{x^2}{x^{1/2}} - \frac{1}{x^{1/2}}}{\frac{x^{1/2}}{x^{1/2}} - \frac{3}{x^{1/2}}}$

$= \lim_{x\to\infty} \frac{\underbrace{x^{3/2}}_{\to\infty} - \underbrace{\frac{1}{x^{1/2}}}_{\to 0}}{\underbrace{1}_{\to 1} - \underbrace{\frac{3}{x^{1/2}}}_{\to 0}} = \boxed{\infty}$

**4 v.** $\lim_{x\to\infty} \sqrt{4-\frac{1}{x}} = \lim_{x\to\infty} \left(4-\frac{1}{x}\right)^{1/2}$

If $\lim_{x\to\infty} \left(4-\frac{1}{x}\right) = L$, then $\lim_{x\to\infty} \left(4-\frac{1}{x}\right)^n = L^n$ by rule 5 on pg. 152.

Thus, since $\lim_{x\to\infty} \left(4-\frac{1}{x}\right) = 4$,

$\lim_{x\to\infty} \left(4-\frac{1}{x}\right)^{1/2} = 4^{1/2} = \boxed{2}$.

**4 vi.** $\lim_{x\to\infty} \sqrt{x^2+1} - x$.

This problem is more challenging.

Multiply by $\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$. So, $\lim_{x\to\infty} \sqrt{x^2+1} - x$

$= \lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right)\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right)$

$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}+x}$

$= \lim_{x\to\infty} \frac{(x^2+1) - x^2}{\sqrt{x^2+1}+x}$ since $(a-b)(a+b) = a^2-b^2$

$= \lim_{x\to\infty} \frac{1 \to 1}{\underbrace{\sqrt{x^2+1}}_{\to\infty} + \underbrace{x}_{\to\infty}} = \boxed{0}$.

5 i. $f(x) = -2x^3 + 3x^2$

Because $f$ is continuous on the three intervals ($[-2,0]$, $[0,2]$, and $[-2,2]$) and because each interval is closed and bounded, by the Extreme Value Theorem, we are guaranteed to have an absolute minimum and an absolute maximum on each interval.

$f'(x) = -6x^2 + 6x$

Find critical values

$-6x^2 + 6x = 0$

$-6x(x-1) = 0$

$x=0$, $x=1$ are the critical values

Now check the endpoints of each interval and the critical values inside each interval.

$[-2,0]$: Check $-2, 0$

$f(-2) = 28$

$f(0) = 0$

The absolute minimum of $f$ in $[-2,0]$ is $\boxed{0}$ and the absolute maximum in $[-2,0]$ is $\boxed{28}$.

$[0,2]$: Check $0, 1, 2$

$f(0) = 0$

$f(1) = 1$

$f(2) = -4$

The absolute minimum of $f$ in $[0,2]$ is $\boxed{-4}$ and the absolute maximum in $[0,2]$ is $\boxed{1}$.

$[-2,2]$: Check $-2, 0, 1, 2$

$f(-2) = 28$

$f(0) = 0$

$f(1) = 1$

$f(2) = -4$

The absolute minimum of $f$ in $[-2,2]$ is $-4$ and the absolute maximum in $[-2,2]$ is 28.

**5 ii.** $f(x) = 3 - x + x^2$ on $(-\infty, \infty)$

Notice that $f$ is continuous on $(-\infty,\infty)$.

$f'(x) = -1 + 2x$

Find critical values

$$-1+2x=0$$
$$2x=1$$
$$x=\tfrac{1}{2}$$

Let's check if this is the location of a maximum using the first derivative test

| | $(-\infty, \frac{1}{2})$ | $(\frac{1}{2}, \infty)$ |
|---|---|---|
| | $f'(x)<0$ | $f'(x)>0$ |
| | ↘ | ↗ |

There is a relative minimum at $x = \frac{1}{2}$.

Because $f$ is continuous and only has one critical point, a minimum, it must be the absolute minimum.

So, the absolute minimum is

$f(\frac{1}{2}) = \boxed{2.75}$

Now notice $\lim_{x\to\infty} 3-x+x^2$

$= \lim_{x\to\infty} x^2(\frac{3}{x^2} - \frac{1}{x} + 1) = \infty$ and

$\lim_{x\to-\infty} 3-x+x^2 = \lim_{x\to-\infty} x^2(\frac{3}{x^2} - \frac{1}{x} + 1) = \infty$

so the absolute maximum $\boxed{\text{does not exist}}$

**5 iii.** $f(x) = 3x + x^{-3}$ on $(0,3]$

Notice that $\lim_{x\to0^+} 3x+x^{-3}$

$= \lim_{x\to0^+} 3x + \frac{1}{x^3} = \infty.$

So, $f$ will not have an absolute maximum on $(0,3]$.

Now find any critical values.

$f'(x) = 3 - 3x^{-4}$

$$3 - \frac{3}{x^4} = 0$$
$$3 = \frac{3}{x^4}$$
$$3x^4 = 3$$
$$x^4 = 1$$
$$x=1,\ x=-1$$

Only $x=1$ is in $(0,3]$.

| | $(0,1)$ | $(1,3]$ |
|---|---|---|
| | $f'(x)<0$ | $f'(x)>0$ |
| | ↘ | ↗ |

So, we have a relative min at $x=1$.

Since $f$ decreases from 0 to 1 and increases from 1 to 3, this will be the absolute min

Thus, the absolute min of $f$ on $(0,3]$ is $f(1) = 3+1 = \boxed{4}$ and the absolute max $\boxed{\text{does not exist}}$

5 iv. $R(x) = -x^3 + 75$

We will look at the interval $[0, \infty)$ because we only care about producing a nonnegative number of goods.

$R'(x) = -3x^2 + 75$

Find critical values

$-3x^2 + 75 = 0$

$3x^2 = 75$

$x^2 = 25$

$x = \pm 5$

Only $x = 5$ is in the interval $[0, \infty)$.

Check if 5 is the location of a relative max.

Second derivative test: $R''(x) = -6x$

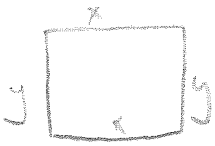
$R''(5) = -6 \cdot 5 = -30 < 0$

So 5 is the location of a max.

Because it is the only critical value in $[0, \infty)$, it is the absolute max.

So, revenue is maximized when production is 5.

5 v. Call the length of the rectangle x and the width y




We know $xy = 20{,}000$.

Call the cost per foot of three sides C so the cost of the fourth side is 3C. Assume $C > 0$.

Thus, the cost of the fence is

$3cy + Cx + Cy + Cx$

$= 4cy + 2cx$

Because $xy = 20{,}000$

$x = \dfrac{20{,}000}{y}$

So the cost equation is

$f(y) = 4cy + 2c \cdot \dfrac{20{,}000}{y}$

$= 4cy + \dfrac{40{,}000c}{y}$

Also, we want the sides to have positive length, so y is in $(0, \infty)$.

Now find the minimum.

$f'(y) = 4c - 40000cy^{-2}$

$= 4c - \dfrac{40000c}{y^2}$

Find critical values

$4c - \dfrac{40{,}000c}{y^2} = 0$

$4\cancel{c} = \dfrac{40000\cancel{c}}{y^2}$

(5 v continued)

$$4 = \frac{40{,}000}{y^2}$$

$$4y^2 = 40{,}000$$

$$y^2 = 10{,}000$$

$$y = \pm 100$$

Only $y=100$ is in $(0,\infty)$

Check that this is a min.

$$f''(y) = (-2)(-40000y^{-3})$$

$$= \frac{80000}{y^3}$$

$f''(100) = \frac{80000}{100^3} > 0$, so 100 is the location of a min.

So $x = \frac{20{,}000}{100} = 200$.

Thus, the dimensions are 200 ft by 100 ft (where the side that costs three times as much is 100 ft).

5 Let $x$ be the number of \$50 increases in rent per day

Then rent $= 300 + 50x$

and the number of occupied units will be $10 - x$.

We need $10 - x \geq 0$, so $x \leq 10$.

Also, we have $x \geq 0$, then $x$ is in $[0, 10]$.

(5 vi continued)

The revenue is the rent times the number of occupied units so revenue is $(300+50x)(10-x)$.

Call revenue $R(x)$.

$$R(x) = (300+50x)(10-x)$$

$$= 3000 + 200x - 50x^2$$

$$R'(x) = 200 - 100x$$

$$200 - 100x = 0$$

$$100x = 200$$

$$x = 2$$

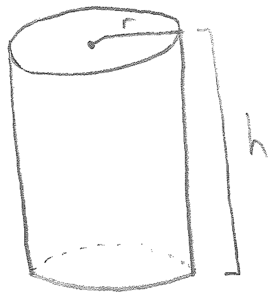
Check that this is a max.

$$R''(x) = -100$$

$R''(2) = -100 < 0$ so max.

Thus, the rent that should be charged to maximize revenue is $300 + 50 \cdot 2 = 300 + 100 = \boxed{\$400}$.

5 vii.




Assume the cylinder has radius r and height h.

Then the volume of the cylinder is $\pi r^2 h$ (This is the area of the top times the height).

We know $\pi r^2 h = 16$.

In order to find the most economical proportions, we want to minimize the amount of material it would take to build the cylinder (this is the same quantity as surface area).

The top has area $\pi r^2$ and the bottom has area $\pi r^2$.

Notice that the side has height h and width equal to the diameter of the top. ( 

) The diameter is $2\pi r$, so the area of the side is $2\pi r h$.

Thus, the amount of material (or surface area) is

$$\pi r^2 + \pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r h.$$

We want this in terms of just one variable so we solve $\pi r^2 h = 16$ for h. Then $h = \frac{16}{\pi r^2}$, so the surface area is

$$2\pi r^2 + 2\pi r \cdot \frac{16}{\pi r^2} = 2\pi r^2 + \frac{32}{r}.$$

Call this $f(r)$.

Notice that r is in $(0, \infty)$.

$f(r) = 2\pi r^2 + \frac{32}{r} = 2\pi r^2 + 32 r^{-1}$ so

$f'(r) = 4\pi r - 32 r^{-2}$

(5 vii. continued)

$$4\pi r - 32r^{-2} = 0$$

$$4\pi r = 32r^{-2}$$

$$4\pi r = \frac{32}{r^2}$$

$$4\pi r^3 = 32$$

$$\pi r^3 = 8$$

$$r^3 = \frac{8}{\pi}$$

$$r = \sqrt[3]{\frac{8}{\pi}}$$

$$r = \frac{2}{\sqrt[3]{\pi}}$$

$\frac{2}{\sqrt[3]{\pi}} \approx 1.3$. Check this is a minimum.

$$f''(r) = 4\pi - 32(-2)r^{-3}$$

$$= 4\pi + \frac{64}{r^3}$$

$f''\left(\frac{2}{\sqrt[3]{\pi}}\right) > 0$, so minimum

$$h = \frac{16}{\pi\left(\frac{2}{\sqrt[3]{\pi}}\right)^2} = \frac{16}{\frac{4\pi}{\pi^{2/3}}} = \frac{16}{4\pi^{1/3}} = \frac{4}{\sqrt[3]{\pi}}$$

So, the most economical dimensions for a cylinder with volume 16 cubic inches is height $\frac{4}{\sqrt[3]{\pi}}$ inches and radius $\frac{2}{\sqrt[3]{\pi}}$ inches.